Hyperbolic groups with few linear representations

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Abstract

In this paper we are going to develop an unpublished argument of William Thurston that shows the existence of hyperbolic groups with no non-trivial representations to linear groups over any ring, for a given dimension. His approach studies the space of representations of any non-elementary hyperbolic group G to conclude that there is a hyperbolic quotient with such property.

1 Introduction

Let us begin with the statement of the main result.

Theorem 1. *Let G be a non-elementary hyperbolic group and n a natural number. Then there exists a quotient group G' of G such that:*

1. *G' is hyperbolic*
2. *Given any commutative ring R, there is no non-trivial representation of G' to $GL_n(R)$.*

Observe that we can consider R to be finitely generated as a $\mathbb{Z}$-algebra. Indeed, hyperbolic groups are finitely generated, hence any representation can be considered on the subring generated by the matrix entries of the generators of the group.

There is an argument from Priyam Patel and Ben McReynolds that solves the problem using super-rigidity of cocompact lattices of $Sp(n,1)$ when we consider representations with coefficients over fields.

First we are going to give an appropriate definition and topology to the space of representations, then using the Cantor-Bendixson rank we are going to reduce the problem to "taking care of finitely many representations", where the proof of our main result ends.

Acknowledgments

I would like to thank my advisor Ian Agol for exposing the argument to me, and Matthew Harrison-Trainor for our talk about ordinals and the Cantor-Bendixson rank. The ideas on this article belong to William Thurston, this work is dedicated to his memory.

2 Space of Representations

Let's fix from now on a finite presentation of $G = \langle X \mid S\rangle$ where all the elements of S have positive exponents. This can be achieved by adding an element A to the generators and aA to the relations for each element $a \in X$. Then replace a^{-1} by A in every word.

Definition 1. *Let $G = \langle X|S\rangle$ be a finitely presented group. We denote by $\mathcal{C}(G)$ (or simply $\mathcal{C}$ when there is not ambiguity) the set*

$$\{H \triangleleft G \mid \textit{there is a ring } R \textit{ and a representation } \rho : G \to GL_n(R) \textit{ such that } \ker \rho = H\}$$

equipped with the Gromov-Hausdorff topology of their assigned Cayley graphs

$$Cayley(G/H) = Cayley(G)/H$$

marked at the identity.

Observe that $\mathcal{C}$ has always one element corresponding to the trivial representations.

Lemma 1. *Let G be a finitely presented group. Then:*

1. *$\mathcal{C}(G)$ is sequentially compact.*
2. *$\mathcal{C}(G)$ is countable.*

Proof. 1. Let $\{H_m\}$ be a sequence in $\mathcal{C}$. Since the Cayley graphs of all the quotients come from the generators of G, we have a bound on the number of vertices and edges of a fix radius (say r) ball of any Cayley graphs. Because this leads to only finitely many models of the r-ball, then any sequence has a subsequence where the r-ball of the Cayley graphs are all the same, for a given r. By Cantor's diagonal argument, we have a subsequence H_{m_k} where the r-ball of the corresponding Cayley graphs of $\{H_{m_k}, k \geq r\}$ are all the same.

Rename the subsequence back to $\{H_m\}$. The graph limit (say $\mathcal{H}$) in the Gromov-Hausdorff topology is clear now, is the increasing union of all the constant r-balls (which agree between them). Then the subgroup H of G consisting of elements that belong to all H_m except for finitely many values of m is the subgroup associated to $\mathcal{H}$. So all that remains to proof that this limit belongs to $\mathcal{C}$ is that H is the kernel of some representation on the matrix group of a ring.

Look first to our current representations, $\{\rho_m\}$. We want to construct a representation ρ such that $x \in G$ is trivial in ρ if and only if x is trivial in all ρ_m, except for maybe finitely many values of m. So we need a ring R where the elements are "collections of elements of R_m, where collections are identified if they are equal except for a finite set". Then define R as $\prod_{k=1}^{\infty} R_k/I$, where I is the ideal of all the elements with finite support. Define ρ as the composition of $\prod_{k=1}^{\infty} \rho_k$ with $GL_n(\prod_{k=1}^{\infty} R_k) \to GL_n(R)$(which comes from the quotient). Then the kernel of ρ are all the elements x that are trivial for all ρ_m, except for finitely many values of m.

2. Consider any ring R. We want to find out restrictions for elements of $GL_n(R)$ that can we assign to the generating set X of G such that this choice can be extended to a homomorphism with kernel a given $H \triangleleft G$.
   We know that the elements of S must be trivial in $GL_n(R)$ in order to define a homomorphism . Hence any such choice of elements for X must have matrix entries satisfying a finite number of polynomials equations with integer coefficients (each coming from computing the corresponding matrices for elements of S from the matrices assigned to X and thanks to only having positive power in S). Analogously, being trivial in H adds a countable system of polynomial equations with integer coefficients (each elements of H can be presented as a word with only positive exponents). By the Hilbert basis theorem, only finitely many of them generate rest of them in $\mathbb{Z}[X]$. Consider then finitely many elements (name this set Y) of H such that the set of polynomials that they define (together with the ones that S generates) includes this finite set of polynomials. This finite set does not depend on the ring considered. Hence choosing matrix entries for X that satisfy this finite equations define a representation on $GL_n(R)$ that is trivial in H, and any such representation can be realized by such a choice. Then if H is actually the kernel of some representation, it is the minimal kernel that we can obtain from the equations given by Y when we go over solutions of any ring. Hence the map:

$$\{Y \subseteq G \mid |Y| < \infty \text{ and defines a minimal kernel}\} \to \mathcal{C}$$

   that sends Y to element corresponding the minimal kernel of the representations satisfying the polynomial equations given by Y and S, is suryective. Then the enumerability of $\mathcal{C}$ follows.

□

Given that a base for the Gromov-Hausdorff topology are set of graphs that coincide with a given graph in a r-ball, we have that in particular $\mathcal{C}$ is a second-countable space. It is also a normal space, given a closed set $\mathcal{F}$ and an element $H \notin \mathcal{F}$, there is r such that the r-ball of the Cayley graph corresponding to H is different from any r-ball coming from $\mathcal{F}$ (if not H will be a limit point of $\mathcal{F}$). Then the open sets given by the r-balls that appear in $\mathcal{F}$ and the r-ball of H are disjoint. In particular, we have:

Corollary 1. *$\mathcal{C}$ is compact.*

3 Cantor-Bendixson rank

Now we are going to define the Cantor-Bendixson rank for $\mathcal{C} = \mathcal{C}^0$. The set of discrete points of $\mathcal{C}^1$ (say $\mathcal{C}_0$) is an open non-empty set. Indeed, since $\mathcal{C}$ is countable and non-empty, it can not be perfect, hence $\mathcal{C}_0 \neq \emptyset$ (openness is obvious). Then $\mathcal{C}^1 = \mathcal{C} \setminus \mathcal{C}_0$ is also compact and countable. If it's non-empty we can take now its set of discrete points, $\mathcal{C}_1$. As we see, this inductive labeling classifies how "discrete" are the points of a countable and compact space (discrete points are labeled by 0, discrete limit points are labeled by 1 and so on). Since there is no reason whatsoever why this induction will label all the elements of $\mathcal{C}$, we are going to use ordinals as our labels and transfinite induction to claim that every element is labeled.

The previous paragraph shows that if we have use the α label (where α is an ordinal) and the remaining close set is non-empty, we can continue to the successor ordinal $\alpha + 1$. The limit ordinal case is similar: let λ be a limit ordinal such that all $\alpha < \lambda$ had been used. Name $\bigcap\limits_{\alpha<\lambda} C^\alpha$ by C^λ (which is closed) and its set of discrete points by $\mathcal{C}_\lambda$ if it's non-empty.

Then there must been an ordinal α with $\mathcal{C}^\alpha = \emptyset$, called the Cantor-Bendixson rank of our set. If not, transfinite induction shows us that every ordinal will be used as a label, but $\mathcal{C}$ is countable. Hence, we even now that only countable ordinals are going to be used.

Claim: α is not a limit ordinal.

Assume that α is a limit ordinal. It is a fact that exists a increasing sequence of ordinals that converges to α. Indeed, since α is countable, take a enumeration of all the ordinals less than α and consider the following subsequence: after the m-term, take the next element in the sequence that is greater than this term. This subsequence is increasing, and given $\beta < \alpha$, all the terms that we pick after β (regardless if we pick β or not) are greater than β, hence we have the convergence. Take an element in each $\mathcal{C}*$ for the terms of our sequence, and since $\mathcal{C}$ is compact, we can assume that this sequence converges. By the previous remark, this limit can not belong to $\mathcal{C}^\beta$ for any $\beta < \alpha$. Since this contradicts that $\mathcal{C}^\alpha = \emptyset$, the claim follows.

This implies that there is β with $\mathcal{C}^\beta$ discrete, hence this set is finite and β is our maximal label. Now we are ready to prove the theorem.

4 Proof of Theorem 1

Proof. Consider β the Cantor-Bendixson rank of $\mathcal{C}(G)$. If this set had only one element we are done because G itself doesn't have non-trivial representations. Let $\mathcal{C}^\beta = \{H_1, H_2, \ldots, H_k\}$. None of this subgroups can be G, because its corresponding Cayley graph has only one vertex, hence is an isolated point. Since G is a non-elementary hyperbolic group, there is a result of Gromov [1] saying

Lemma 2 (Gromov)**.** *Let G be a non-elementary hyperbolic subgroup, $E_0 \subset G$ an infinite elementary subgroup and let $C \subset E_0$ be a maximal infinite cyclic subgroup. Denote E_0/mC by E_m. then exists $k = k(G, E_0)$ such that for every $m \geq k$ exists a quotient G' of G that holds:*

1. *G' is non-elementary hyperbolic.*

2. *The image of E_0 under the quotient is isomorphic to E_m.*

3. *Every torsion element of G' is either conjugate to an element of E_0 or it is the image of a torsion element.*

Since infinite order elements are dense in G, we can consider an element x of infinite order that is non-trivial in H_1. Applying Gromov's result for $E_0 = C = \langle x \rangle$ for an appropriate m such that x^m is also non-trivial in H_1. Then x^m is trivial in G'. Observe that there is a natural inclusion $\mathcal{C}(G') \hookrightarrow \mathcal{C}(G)$ given by the quotient map. Since this inclusion is actually an embedding, the maximal Cantor-Bendixson index of G' (say β') is less or equal than β.

Assume than $\beta' = \beta$. Take ρ one representation of G' with Cantor-Bendixson index β'. Then there are some H_i (at least one) such that the diagram:

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\rho_1"] & \\ G' \arrow[r, "\rho"] & GL_n(R_1) \end{tikzcd}$$

commutes. but H_1 can not be one of them because x^m is non-trivial under ρ_1 but is trivial in G'. Then G' has less representations that G with index β, so we after taking successive quotients if needed, we can assume that $\beta' < \beta$.

Now by this process we can construct a sequence of successive hyperbolic quotients that define a descending sequence of ordinals. By the ordinal version of infinite descent, this sequence has to be finite, hence there is a non-elementary hyperbolic quotient group G' of G with $\mathcal{C}(G)$ consisting of only one element, proving the theorem. $\square$

References

[1] M Gromov, *Hyperbolic Groups*. Mathematical Sciences Research Institute Publications. Volume 8, 1987 pp 75-263